Pre-class Warm-up!!!

How would you solve the following equations?

$2xy + x^2 y' = y^2$ d $\quad 2xy + x^2 y' = y^2$

$xy^2 + 3y^2 - x^2 y' = 0$ $\quad xy^2 + 3y^2 - x^2 y' = 0$ b

$xy' + 2y = 6x^2 \sqrt{y}$ e

$xy' + 2y = 6x^2$ c

$y' = \sqrt{(x+y)}$

$y'' = 5 / y^2$ $\quad \dfrac{d^2y}{dt^2} = \dfrac{5}{y^2}$

$2xy \cdot y' + y^2 = 10x$ $\quad 2xy\dfrac{dy}{dx} + y^2 = 10x$

a. by integrating y' = function of x
b. separate the variables
c. as a first order linear equation
d. as a homogeneous equation
e. as a Bernoulli equation
f. make a special substitution
g. reduce the order
h. as an exact equation

In the exam you may use a single sheet (2 sides) of handwritten notes.

True or false?

1. If a system of linear equations has more equations than unknown variables then there is no solution.

2. If a system has fewer equations than variables there is always a solution.

3. If a system has fewer equations than variables and there is at least one solution then there are infinitely many solutions.

Section 3.5. Inverses of matrices

New vocabulary:
- Inverse matrix
- Invertible matrix = non-singular matrix
- Elementary matrix

We learn:
- Formula for the inverse of a 2 x 2 matrix
- How to find the inverse in general using Gauss-Jordan elimination.
- Use of the inverse in solving equations

The hardest thing:
Theorem 7. Properties of nonsingular matrices.

An inverse for a matrix A is a matrix B so that $AB = BA = I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

= a matrix that has an inverse.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Page 185 question 2.

Find A^{-1}; then use A^{-1} to solve the system Ax = b where

$$A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \qquad b = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

i.e. Solve $Ax = b$

(Related question: find a matrix X so that

$$AX = \begin{bmatrix} -1 & 0 & -4 \\ 3 & 2 & 7 \end{bmatrix}$$

Solution: $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}^{-1} = \frac{1}{15-14}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

We solve $Ax = b$ by doing

$A^{-1}Ax = A^{-1}b$

$x = A^{-1}b$

$$x = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 - 21 \\ 2 + 9 \end{bmatrix} = \begin{bmatrix} -26 \\ 11 \end{bmatrix}$$

(This solves $\begin{aligned} 3x_1 + 7x_2 &= -1 \\ 2x_1 + 5x_2 &= 3 \end{aligned}$.)

$$X = A^{-1}AX = A^{-1}\begin{bmatrix} -1 & 0 & -4 \\ 3 & 2 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 & -4 \\ 3 & 2 & 7 \end{bmatrix} = \begin{bmatrix} -26 & -14 & -69 \\ 11 & 6 & 29 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Question:

If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, what is the (1,2) entry of A^{-1}?

a. -3/2

b. -1

c. 3/2 ✓

d. 2

e. None of the above.

The best method to find inverses of 3x3 matrices or larger

Like questions 15-22

Find the inverse of $A = \begin{bmatrix} 2 & 4 & 7 \\ 1 & 2 & 2 \\ -1 & 0 & 5 \end{bmatrix}$

Solution We use Gauss Jordan elimn.

on

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 7 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ -1 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{1 \leftrightarrow 2} \begin{bmatrix} 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 4 & 7 & 1 & 0 & 0 \\ -1 & 0 & 5 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[3 \to 3 + 1]{2 \to 2 - 2\,1} \begin{bmatrix} 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 & -2 & 0 \\ 0 & 2 & 7 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{2 \leftrightarrow 3} \begin{bmatrix} 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 2 & 7 & 0 & 1 & 1 \\ 0 & 0 & 3 & 1 & -2 & 0 \end{bmatrix} \xrightarrow[3 \to \frac{1}{3}3]{2 \to \frac{1}{2}2} \begin{bmatrix} 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & \frac{7}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & 0 \end{bmatrix}$$

$$\xrightarrow[2 \to 2 - \frac{7}{2}3]{1 \to 1 - 2\,3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -\frac{2}{3} & \frac{7}{3} & 0 \\ 0 & 1 & 0 & -\frac{7}{6} & \frac{17}{6} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & 0 \end{array}\right]$$

$$\xrightarrow{1 \to 1 - 2\,2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{3} & -\frac{10}{3} & -1 \\ 0 & 1 & 0 & -\frac{7}{6} & \frac{17}{6} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & 0 \end{array}\right]$$

Conclude $A^{-1} = \frac{1}{6} \begin{bmatrix} 10 & -20 & -6 \\ -7 & 17 & 3 \\ 2 & -4 & 0 \end{bmatrix}$

$$A A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

Why does this work?

Elementary matrices are of 3 kinds, like

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longleftrightarrow \text{swap rows 1 and 2.}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longleftrightarrow \text{multiply row 2 by } 7 \neq 0$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longleftrightarrow \text{add } 3 \cdot (3) \text{ to } (1)$$

Theorem 5. Doing an elementary row operation to a matrix A produces the answer EA, where E is the corresponding elementary matrix.

Example Take $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ -1 & 2 & 0 \end{bmatrix}$

and $(1) \to (1) + 3(3)$

$$\begin{bmatrix} -2 & 7 & -1 \\ 0 & 2 & 1 \\ -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ -1 & 2 & 0 \end{bmatrix}$$

Each step in finding A^{-1} corresponds to left multn by an elementary matrix

Start $\left[\begin{array}{c|ccc} ? & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \end{array}\right] \longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left[\begin{array}{c|c} ? & 1 \\ & 1 \\ & 1 \end{array}\right]$

$$= \left[\begin{array}{c|ccc} ? & 0 & 1 & 0 \\ & 1 & 0 & 0 \\ & 0 & 0 & 1 \end{array}\right] \quad \cdots$$

$$\underbrace{E_n \cdots E_3 E_2 E_1}\left[\begin{array}{c|c} A & I \end{array}\right] = \left[\begin{array}{c|c} I & \text{New} \end{array}\right]$$

$$\underset{\substack{\| \\ \text{New}}}{E_n \cdots E_1} = A^{-1}$$

Theorem. Every invertible matrix can be written as a product of elementary matrices.

Theorems 1 and 3. If A is invertible then A^{-1} is unique.
$(AB)^{-1} = B^{-1}A^{-1}$
Etc

Theorem 7. Let A be an $n \times n$ matrix. The following are equivalent.
a. A is invertible.
b. A is row equivalent to I.
c. $Ax = 0$ has only the trivial solution.
d. For all b, $Ax = b$ has a unique solution.
e. For all b, $Ax = b$ is consistent.